Journal of Global Optimization **27:** 135–148, 2003.

Computation of Signature Symmetric Balanced Realizations

U. HELMKE[1], K. HÜPER[1] and J.B. MOORE[1,2]
[1]*Department of Mathematics, University of Würzburg, Am Hubland, D-97074 Würzburg, Germany (e-mail: helmke@mathematik.uni-wuerzburg.de hueper@mathematik.uni-wuerzburg.de)*
[2]*Department of Systems Engineering, RSISE, The Australian National University, Canberra ACT 0200, Australia (e-mail: John.Moore@anu.edu.au)*

Abstract. A new numerical scheme for computing balancing coordinate transformations for signature symmetric realizations in linear systems theory is presented. The method is closely related to the Jacobi method for diagonalizing symmetric matrices. Here the minimization of the sum of traces of the Gramians by orthogonal and hyperbolic Jacobi-type rotations is considered. Local quadratic convergence of the algorithm is shown.

Key words: Balancing, signature symmetric realizations, cost function, Jacobi method, quadratic convergence

1. Introduction

Since the important work by Glover (1984) [7] and Ober (1987) [13], and despite a growing literature on balanced realizations and model reduction there are only few papers dealing with numerical aspects.

The first such numerical algorithms for balancing have been proposed by Laub *et al.* (1987) [11] and Safonov and Chiang (1989) [14]. There exists an amount of literature on balancing, model reduction techniques, and related algorithms in the area of flexible structures. We mention the monograph by Gawronski (1996) [6]. There is also the monograph by Fortuna *et al.* (1992) on model reduction techniques and algorithms with applications in electrical engineering [5]. For applications of balancing algorithms in circuit synthesis see De Abreu-García *et al.* (1987) [1]. Meanwhile, there are also efforts to parallelize known algorithms for balancing, see the recent work by Benner *et al.* (1999) [3]. All the above mentioned work on algorithms have in common that a full convergence theory is not available.

Numerical gradient flow algorithms for balancing, as presented by Moore and Helmke (1994) [9] and Yan *et al.* (1994) [15] offer an alternative approach with a better convergence theory, but these algorithms are only linearly convergent. In [8] a Jacobi-type algorithm for computing balanced realizations is presented with local quadratic convergence rate. Here we extend the analysis of [8] to balancing of signature symmetric realizations. For existence and uniqueness results on signature symmetric realizations see Youla and Tissi (1966) [16], Anderson and Bitmead (1977) [2], and Byrnes and Duncan (1982) [4]. For recent work on model reduction for state-space symmetric systems see e.g. Liu *et al.* (1998) [12].

It is well konwn that balanced realizations of symmetric transfer functions are signature symmetric. The above mentioned algorithms [11] and [14], however, do not preserve the signature symmetry and they may be sensible to numerical perturbations from the signature symmetric class.

In recent years there is a tremendous interest in structure preserving (matrix) algorithms. The main motivation for this is twofold. If such a method can be constructed it usually (i) leads to reduction in complexity and (ii) often coincidently avoids that in finite arithmetic physically meaningless results are obtained. Translated to our case that means that (i) as the appropriate state space transformation group the Lie group $\mathrm{O}^{+}_{pq}(\mathbb{R})$ of special pseudo-orthogonal transformations is used instead of $\mathrm{GL}_n(\mathbb{R})$. Furthermore, (ii) at any stage of our algorithm the computed transformation corresponds to a signature symmetric realization if one would have started with one.

2. Cost Function Approach to Balancing

In this section at first we briefly review notions and results on balancing and signature symmetric realizations. Given any asymptotically stable linear system (A, B, C), the continuous-time controllability Gramian W_c and the observability Gramian W_o are defined, respectively, by

$$\begin{aligned} W_c &= \int_0^\infty \mathrm{e}^{tA} B B' \mathrm{e}^{tA'} \mathrm{d}\, t, \\ W_o &= \int_0^\infty \mathrm{e}^{tA'} C' C \mathrm{e}^{tA} \mathrm{d}\, t. \end{aligned} \tag{1}$$

Thus, assuming controllability and observability, the Gramians W_c, W_o are symmetric positive definite matrices. Moreover, a linear change of variables in the state space by an invertible state space coordinate transformation T leads to the co- and contravariant transformation law of Gramians as

$$(W_c, W_o) \mapsto \left(T W_c T', (T')^{-1} W_o T^{-1}\right). \tag{2}$$

Let $p, q \in \mathbb{N}_0$ be integers with $p + q = n$, $I_{pq} := \operatorname{diag}(I_p, -I_q)$. A realization $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ is called *signature symmetric* if

$$\begin{aligned} (AI_{pq})' &= AI_{pq}, \\ (CI_{pq})' &= B \end{aligned} \tag{3}$$

holds. Note that every strictly proper symmetric rational $(m \times m)$-transfer function $G(s) = G(s)'$ of McMillan degree n has a minimal signature symmetric realization and any two such minimal signature symmetric realizations are similar by a unique state space similarity transformation $T \in O_{pq}(\mathbb{R})$. The set

$$\mathrm{O}_{pq}(\mathbb{R}) := \{T \in \mathbb{R}^{n\times n} | T I_{pq} T' = I_{pq}\}$$

is the real Lie group of pseudo-orthogonal $(n \times n)$-matrices stabilizing I_{pq} by congruence. The set $\mathrm{O}^+_{pq}(\mathbb{R})$ denotes the identity component of $\mathrm{O}_{pq}(\mathbb{R})$. Here $p-q$ is the Cauchy-Maslov index of $G(s)$, see [2] and [4]. For any stable signature symmetric realization the controllability and observability Gramians satisfy

$$W_o = I_{pq} W_c I_{pq}. \tag{4}$$

As usual, a realization (A, B, C) is called balanced if

$$W_c = W_o = \Sigma = \mathrm{diag}\,(\sigma_1, \ldots , \sigma_n) \tag{5}$$

where the $\sigma_1, \ldots , \sigma_n$ are the Hankel singular values. For simplicity in the sequel we assume that they are pairwise distinct.

LEMMA 2.1. *Let*

$$M(\Sigma) := \{T \Sigma T' \mid T \in \mathrm{O}^+_{pq}(\mathbb{R})\}, \tag{6}$$

with Σ as in (5) assuming pairwise distinct Hankel singular values. Then
(1) $M(\Sigma)$ is a smooth and connected manifold of dimension

$$\dim M(\Sigma) = n(n-1)/2. \tag{7}$$

(2) The tangent space of $M(\Sigma)$ at $X \in M(\Sigma)$ is

$$T_X M(\Sigma) = \left\{ \Psi X + X\Psi' \,\middle|\, \Psi \in \mathfrak{o}_{pq}(\mathbb{R}) \right\}. \tag{8}$$

Proof. Consider the linear algebraic group action

$$\begin{aligned} &\alpha : \mathrm{O}^+_{pq}(\mathbb{R}) \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \\ &\alpha(T, X) := T X T'. \end{aligned} \tag{9}$$

Thus $M(\Sigma)$ is an orbit of the group action α under the connected Lie group $\mathrm{O}^+_{pq}(\mathbb{R})$ and therefore a smooth and connected manifold. Note that the stabilizer subgroup of a point $X \in M(\Sigma)$ is finite and therefore $M(\Sigma)$ is diffeomorphic to $\mathrm{O}^+_{pq}(\mathbb{R})$ which as a pseudo-orthogonal group of order $n = p+q$ has dimension $n(n-1)/2$.

Consider the smooth map

$$\begin{aligned} &\sigma : \mathrm{O}^+_{pq}(\mathbb{R}) \to M(\Sigma), \\ &\sigma(T) := T X T'. \end{aligned} \tag{10}$$

The tangent space of $O_{pq}^{+}(\mathbb{R})$ at the identity I_n is $T_{I_n}O_{pq}^{+}(\mathbb{R}) = \mathfrak{o}_{pq}(\mathbb{R})$. The derivative of σ at I_n is the surjective linear map

$$\begin{aligned} &\mathrm{D}\sigma|_{I_n} : \mathfrak{o}_{pq}(\mathbb{R}) \to T_X M(\Sigma), \\ &\Psi \mapsto \Psi X + X\Psi'. \end{aligned} \tag{11}$$

The result follows. □

Let $N := \operatorname{diag}(\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$ with $0 < \mu_1 < \cdots < \mu_p$ and $0 < \nu_1 < \cdots < \nu_q$. We then consider the smooth cost function

$$\begin{aligned} &f_N : M(\Sigma) \to \mathbb{R}, \\ &f_N(W) := \operatorname{tr}(NW). \end{aligned} \tag{12}$$

This choice is motivated by our previous work on balanced realizations [8], where we studied the smooth function $\operatorname{tr}(N(W_c + W_o))$ with diagonal positive definite N having distinct eigenvalues. Now

$$\begin{aligned} \operatorname{tr}(N(W_c + W_o)) &= \operatorname{tr}(N(W_c + I_{pq}W_cI_{pq})) \\ &= 2\operatorname{tr}(NW_c) \end{aligned}$$

by the above choice of a diagonal N. The following result summarizes the basic properties of the cost function f_N.

THEOREM 2.1. *Let* $N := \operatorname{diag}(\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q)$ *with* $0 < \mu_1 < \cdots < \mu_p$ *and* $0 < \nu_1 < \cdots < \nu_q$. *For the smooth cost function* $f_N : M(\Sigma) \to \mathbb{R}$, *defined by* $f_N(W) := \operatorname{tr}(NW)$, *the following holds true.*

(1) $f_N : M(\Sigma) \to \mathbb{R}$ *has compact sublevel sets and a minimum of* f_N *exists.*

(2) $X \in M(\Sigma)$ *is a critical point for* $f_N : M(\Sigma) \to \mathbb{R}$ *if and only if* X *is diagonal.*

(3) The global minimum is unique and it is characterized by $X = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$, *where* $\sigma_1 > \cdots > \sigma_p$ *and* $\sigma_{p+1} > \cdots > \sigma_n$ *holds.*

(4) The Hessian of the function f_N *at a critical point is nondegenerate.*

Proof. (1) follows from Theorem 6.3.5 and Lemma 6.4.1 in [9] if one restricts the $\mathrm{GL}_n(\mathbb{R})$ action in [9] to the above $O_{pq}^{+}(\mathbb{R})$-action. The derivative of f_N at X is the linear function assigning to $\Psi X + X\Psi'$ the value $2\operatorname{tr}(N\Psi X)$. Now we partition the matrices

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \tag{13}$$

and

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}' & X_{22} \end{bmatrix} \tag{14}$$

into sub matrices according to

$$N = \operatorname{diag}(N_\mu, N_\nu)$$

with

$$N_\mu := \mathrm{diag}\,(\mu_1, \ldots, \mu_p) \quad \text{and} \quad N_\nu := \mathrm{diag}\,(\nu_1, \ldots, \nu_q).$$

Note that for $\Psi \in \mathfrak{o}_{pq}(\mathbb{R})$ holds $\Psi_{11} = -\Psi'_{11}$, $\Psi_{22} = -\Psi'_{22}$, $\Psi_{12} = \Psi'_{21}$. Thus the derivative is zero if and only if

(i) $[X_{11}, N_\mu] = 0$

(ii) $[X_{22}, N_\nu] = 0$,

(iii) $X'_{12}N_\mu + N_\nu X'_{12} = 0$.

By assumption on the matrix N conditions (i) and (ii) force X_{11} and X_{22} to be diagonal, respectively, whereas (iii) demands $X_{12} = 0$. This proves (2).

Next we compute the Hessian of f_N at a critical point

$$X = \mathrm{diag}\,(y_1, \ldots, y_p, z_1, \ldots, z_q).$$

After cumbersome block matrix computations one arrives at

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} f_N\left(\mathrm{e}^{t\Psi} X \mathrm{e}^{t\Psi'}\right)\bigg|_{t=0} &= 2\mathrm{tr}\,(\Psi^2 XN + \Psi X \Psi' N) \\ = \sum_{1\leqslant i<j\leqslant p} \left(\psi_{ij}^{(11)}\right)^2 (y_j - y_i)(\mu_i - \mu_j) \\ + \sum_{1\leqslant i<j\leqslant q} \left(\psi_{ij}^{(22)}\right)^2 (z_j - z_i)(\nu_i - \nu_j) \\ + \sum_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant q}} \left(\psi_{ij}^{(12)}\right)^2 (y_i + z_j)(\mu_i + \nu_j). \end{aligned} \tag{15}$$

Here $\psi_{kl}^{(rs)}$ denotes the kl-entry of the rs-sub matrix Ψ_{rs} of Ψ, see (13). Therefore by assumption on the entries of the matrix N the Hessian is positive definite if and only if the two subsets $\{y_1, \ldots, y_p\}$ and $\{z_1, \ldots, z_q\}$ of the Hankel singular values are ordered on the diagonal of the critical point X as required. Moreover, because the Hankel singular values are pairwise distinct the quadratic form (15) is nondegenerate at a critical point.

It remains to show that the global minimum is unique. Let $V, W \in M(\Sigma)$ be two different critical points. As a consequence of Theorem 6.3.5 in [9] they differ in how the Hankel singular values are ordered on the diagonal, i.e., there exists an orthogonal permutation matrix P such that $W = PVP'$ holds. Now

$$\mathrm{O}^+_{pq}(\mathbb{R}) \cap \mathrm{SO}_n(\mathbb{R}) = \mathrm{SO}_p(\mathbb{R}) \times \mathrm{SO}_q(\mathbb{R})$$

with $\mathrm{SO}_r(\mathbb{R}) := \{T \in \mathbb{R}^{r\times r} | TT' = I_r, \det(T) = 1\}$. Therefore the permutation matrix has to be block diagonal,

$$P = \mathrm{diag}\,(P_1, P_2)$$

with permutation matrices P_1 and P_2 of order p, respectively q. It follows that the global minimum $X = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, with $\sigma_1 > \cdots > \sigma_p$ and $\sigma_{p+1} > \cdots > \sigma_n$ is unique, moreover, there is no further local minimum, because for any other critical point than X either the sum in the second row of (15) or the sum in the third row of (15) has at least one positive summand, which means that the Hessian is no longer positive definite but indefinite. This proves (3) and (4). □

3. Jacobi-type Algorithm for Balancing

The constraint set for our cost function $f_N : M(\Sigma) \to \mathbb{R}$ is the Lie group $\mathrm{O}^+_{p,q}(\mathbb{R})$ with Lie algebra $\mathfrak{o}_{pq}(\mathbb{R})$. We choose a basis of $\mathfrak{o}_{pq}(\mathbb{R})$ as

$$\Omega_{ij} := e_j e_i' - e_i e_j' \tag{16}$$

where $1 \leqslant i < j \leqslant p$ or $p+1 \leqslant i < j \leqslant n$ holds and

$$\Omega_{kl} := e_l e_k' + e_k e_l' \tag{17}$$

where $1 \leqslant k \leqslant p < l \leqslant n$ holds. These basis elements are defined via the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Thus $\exp(t\Omega_{ij})$ is an orthogonal rotation with $(i, j)-$th sub matrix

$$\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \tag{18}$$

and $\exp(t\Omega_{kl})$ is a hyperbolic rotation with $(k, l)-$th sub matrix

$$\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}. \tag{19}$$

Let N as in Theorem 2.1 above and let W be symmetric positive definite. Consider the smooth function

$$\begin{aligned} &\phi : \mathbb{R} \to \mathbb{R}, \\ &\phi(t) := \operatorname{tr}\left(N \mathrm{e}^{t\Omega} W \mathrm{e}^{t\Omega'}\right) \end{aligned} \tag{20}$$

where Ω denotes a fixed element of the above basis of $\mathfrak{o}_{pq}(\mathbb{R})$. We have

LEMMA 3.1.

(1) *For $\Omega = \Omega_{kl} = (\Omega_{kl})'$ as in (17) the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by (20) is proper and bounded from below.*

(2) *A minimum*

$$t_\Omega := \arg\min_{t \in \mathbb{R}} \phi(t) \in \mathbb{R} \tag{21}$$

exists for all $\Omega = \Omega_{ij} = -(\Omega_{ij})'$ where $1 \leqslant i < j \leqslant p$ or *$p+1 \leqslant i < j \leqslant n$ holds, and exists as well for all $\Omega = \Omega_{kl} = (\Omega_{kl})'$ where $1 \leqslant k \leqslant p < l \leqslant n$ holds.*

Proof. See Lemma 3 in [8]. □

We now formulate a Jacobi-type algorithm to minimize $f_N : M(\Sigma) \to \mathbb{R}$, $f_N(W) = \operatorname{tr}(NW)$. Using an arbitrary ordering of the above basis elements we denote by $e^{t\Omega_1}, \ldots, e^{t\Omega_{n(n-1)/2}}$ the $n(n-1)/2$ different basic transformations defined by (16) and (17). The proposed algorithm for minimizing the smooth cost function f_N then consists of a recursive application of so-called sweep operations. Let for $1 \leqslant i \leqslant n(n-1)/2$

$$\begin{aligned} &r_i : M(\Sigma) \times \mathbb{R} \to M(\Sigma), \\ &r_i(W, t) := e^{t\Omega_i} W e^{t\Omega_i'}. \end{aligned} \tag{22}$$

ALGORITHM 3.1. (Jacobi Sweep). *Define*

$$\begin{aligned} W_k^{(0)} &:= W \\ W_k^{(1)} &:= r_1\Big(W_k^{(0)}, t_*^{(1)}\Big) \\ W_k^{(2)} &:= r_2\Big(W_k^{(1)}, t_*^{(2)}\Big) \\ &\vdots \\ W_k^{(n(n-1)/2)} &:= r_{n(n-1)/2}\Big(W_k^{(n(n-1)/2-1)}, t_*^{(n(n-1)/2)}\Big) \end{aligned}$$

where for $i = 1, \ldots, n(n-1)/2$

$$t_*^{(i)} := \arg\min_{t \in \mathbb{R}} \operatorname{tr}\left(N e^{t\Omega_i} W_k^{(i-1)} e^{t\Omega_i'}\right)$$

if

$$\operatorname{tr}\left(N e^{t\Omega_i} W_k^{(i-1)} e^{t\Omega_i'}\right) \not\equiv \operatorname{tr}\Big(N W_k^{(i-1)}\Big),$$

and

$$t_*^{(i)} := 0$$

otherwise.

Thus $W_k^{(i)}$ is recursively defined as a global minimum of the smooth function $f_N : M(\Sigma) \to \mathbb{R}$, when restricted to the $i-$th smooth curve

$$\begin{aligned} &\gamma_i : \mathbb{R} \to M(\Sigma), \\ &t \mapsto e^{t\Omega_i} W_k^{(i-1)} e^{t\Omega_i'} \end{aligned} \tag{23}$$

containing $\gamma_i(0) = W_k^{(i-1)}$. The algorithm then consists of the iteration of sweeps as discribed in Algorithm 3.2.

ALGORITHM 3.2. (Jacobi-type Algorithm).
(1) For $k \in \{0, 1, 2, 3, \dots\}$ *let* $W_0, \dots, W_k \in M(\Sigma)$ *be given.*
(2) Define the finite recursive sequence

$$W_k^{(1)}, \dots, W_k^{(n(n-1)/2)}$$

as above (Jacobi Sweep).
(3) Set $W_{k+1} := W_k^{(n(n-1)/2)}$*. Proceed with the next sweep applied to* W_{k+1}.

We continue to study the smoothness properties of a sweep, i.e., of one recursion step. A sweep

$$s : M(\Sigma) \to M(\Sigma) \tag{24}$$

of the Jacobi algorithm consists of $n(n-1)/2$ transformations

$$s(W) := (r_{n(n-1)/2} \circ \cdots \circ r_1)(W). \tag{25}$$

By abuse of our former notation here

$$r_i(W) := \mathrm{e}^{t_i(W)\Omega_i} W \mathrm{e}^{t_i(W)\Omega_i'} \tag{26}$$

and $t_i(W)$ denotes a global minimum of the function

$$t \mapsto \varphi(W, t) := \operatorname{tr}\left(N\mathrm{e}^{t\Omega_i} W \mathrm{e}^{t\Omega_i'}\right)$$

with $1 \leqslant i \leqslant n(n-1)/2$. A simple calculation shows that $t_i(W)$ can be given explicitly, see Equations (25) and (27) in [8] for details. We get

LEMMA 3.2. *The map* $r_i : M(\Sigma) \to M(\Sigma)$*, defined by (26) is smooth on an open neighborhood of any critical point* X *of the function* f_N *defined by (12).* □

Proof. For skew-symmetric $\Omega_{ij} = e_j e_i' - e_i e_j'$ the result is a direct consequence of Lemma 5.1.1 in [10]. For symmetric $\Omega_{kl} = e_l e_k' + e_k e_l'$ note that there is an explicit formula for the unique global minimizer, namely

$$t_{kl}(W) = \frac{1}{2} \tanh^{-1} \frac{2w_{kl}}{w_{kk} + w_{ll}}$$

which by the positive definiteness of W is even globally smooth, see [8] Lemma 5 for details. □

The convergence properties of the above algorithm are established by the following theorem which is the main result of this paper.

THEOREM 3.1. *Let* $X_0 := W_c = I_{pq} W_o I_{pq}$ *denote the controllability Gramian of a stable signature symmetric minimal system* (A, B, C)*. Let the matrix* N *as in Theorem 2.1. Let* (W_k)*,* $k = 0, 1, 2, \dots$*, denote the sequence of symmetric* *positive definite matrices generated by the Jacobi-type algorithm 3.2. Assume that global convergence holds, i.e.,*

$$\lim_{k\to\infty} W_k = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$$

with $\sigma_1 > \cdots > \sigma_p$ *and* $\sigma_{p+1} > \cdots > \sigma_n$.

Then convergence of the algorithm is locally quadratically fast.

Proof. Whereas pointwise convergence is strongly indicated by simulations, we do not have a proof for that at present.

For local quadratic convergence the argument is as follows. By Lemma 3.2 the map $r_i : M(\Sigma) \to M(\Sigma)$, i.e. one partial iteration step, is smooth on an open neighborhood U of the global minimum W_∞. Consequently, a whole iteration step, i.e. a sweep, $s : M(\Sigma) \to M(\Sigma)$, defined by (25), is smooth on U. The derivative of a partial iteration step (26) acting on $\Xi \in T_{W_\infty} M(\Sigma)$ is

$$\begin{aligned} \mathrm{D}r_i(W_\infty)\cdot\Xi &= \Xi + (\Omega_i W_\infty + W_\infty\Omega_i')\cdot \mathrm{D}t(W_\infty)\cdot\Xi \\ &= \Xi + (\Omega_i W_\infty + W_\infty\Omega_i')\frac{-\operatorname{tr} N(\Omega_i\Xi + \Xi\Omega_i')}{2\operatorname{tr} N(\Omega_i^2 W_\infty + \Omega_i W_\infty \Omega_i')} \\ &= \Xi - \psi_i(\Omega_i W_\infty + W_\infty\Omega_i') \end{aligned} \tag{27}$$

where we have used the implicit function theorem to compute the derivative of $t : M(\Sigma) \to \mathbb{R}$. Furthermore, we used for elements Ξ of the tangent space $T_{W_\infty} M(\Sigma)$ the representation $\Xi = \Psi W_\infty + W_\infty \Psi'$ with $\Psi \in \mathfrak{o}_{pq}(\mathbb{R})$, and the expansion $\Psi = \sum_{i=1}^{n(n-1)/2} \psi_i \Omega_i$. Now by the chain rule the derivative of a whole iteration step s vanishes at the global minimum W_∞ identically

$$\mathrm{D}s(W_\infty) \equiv 0. \tag{28}$$

Choose open, relatively compact neighborhoods $U, V \subset M(\Sigma)$ of W_∞ such that $s(U) \subset V$ and U, V are diffeomorphic to open subsets of $\mathbb{R}^{n(n-1)/2}$. Without loss of generality we may then assume that U, V are open, bounded subsets of $\mathbb{R}^{n(n-1)/2}$. Reformulating everything in local coordinates, from Taylor's theorem, using (28), we obtain

$$\|s(W_k) - W_\infty\| \leqslant \sup_{W\in\overline{U}} \|\mathrm{D}^2 s(W)\| \cdot \|W_k - W_\infty\|^2. \tag{29}$$

Thus the sequence (W_k) converges quadratically fast to W_∞. □

4. Computer Code

It is easy to present computer codes for the algorithms. The first one, Algorithm 4.1, defines a function that diagonalizes symmetric $2 \times 2-$matrices by orthogonal transformations such that the eigenvalues appear in decreasing order. The second function determines a hyperbolic congruence transformation that minimizes the trace of a symmetric positive definite $2 \times 2-$matrix. Algorithm 4.3 presents a complete Jacobi scheme for computing signature symmetric balanced realizations with sorted Hankel singular values. In Algorithm 4.3 a sweep is built up by three subsweeps. The first one operating on the first p rows and columns by orthogonal transformations, followed by the second subsweep which operates by hyperbolic transformations, and the last one operating on the last q rows and columns again by orthogonal transformations. MATLAB-like algorithmic language and notation is used. The matrices T_{kl} appearing below differ from the identity matrix only by the $(kl)-$th submatrix $\begin{bmatrix} t_{kk} & t_{kl} \\ t_{lk} & t_{ll} \end{bmatrix}$ with $1 \leqslant k, l \leqslant n$. Accumulation of the congruence transformations is possible, but is not explicitly formulated in the algorithms.

ALGORITHM 4.1. *Given $X = X' \in \mathbb{R}^{n \times n}$ and $k, l \in \mathbb{N}$ with $1 \leqslant k < n$ and $k + 1 \leqslant l \leqslant n$, a $(cos, sin)-$pair is computed such that if $\widetilde{X} = T_{kl} X T_{kl}'$ then $\tilde{x}_{kl} = 0$ and $\tilde{x}_{kk} \geqslant \tilde{x}_{ll}$.*

function: $(t_{kk}, t_{kl}, t_{lk}, t_{ll}) = \mathbf{ortho}\ (X, k, l)$
  if $x_{kl} \neq 0$
    if $x_{ll} \neq x_{kk}$

$$\tau = \frac{x_{ll} - x_{kk}}{2x_{kl}}$$

$$tan = \frac{\text{sign}\,(\tau)}{|\tau| + \sqrt{1 + \tau^2}}$$

$$cos = \frac{1}{\sqrt{1 + tan^2}}$$

$$sin = tan \cdot cos$$

    else

$$cos = \frac{1}{2}\sqrt{2}$$

$$sin = -\text{sign}\,(x_{kl}) \cdot cos$$

    end
  else
    $cos = 1$
    $sin = 0$
  end
  if $x_{kk} \geqslant x_{ll}$
      $t_{kk} = cos$
      $t_{kl} = -sin$
      $t_{lk} = sin$

$t_{ll} = cos$
else
$t_{kk} = sin$
$t_{kl} = cos$
$t_{lk} = -cos$
$t_{ll} = sin$
end
end ortho

ALGORITHM 4.2. *Given a symmetric positive definite matrix* $X \in \mathbb{R}^{n\times n}$ *and* $k, l \in \mathbb{N}$ *with* $1 \leqslant k < n$ *and* $k + 1 \leqslant l \leqslant n$, *a* $(cosh, sinh)-$*pair is computed such that if* $\widetilde{X} = T_{kl} X T'_{kl}$ *then* $\tilde{x}_{kk} + \tilde{x}_{ll}$ *is minimized.*

function: $(t_{kk}, t_{kl}, t_{lk}, t_{ll}) =$ **hypo** (X, k, l)
$a = -2x_{kl}$
$b = x_{kk} + x_{ll}$
$cosh = \dfrac{\sqrt{b+a} + \sqrt{b-a}}{2\sqrt[4]{b^2 - a^2}}$
$sinh = \dfrac{\sqrt{b+a} - \sqrt{b-a}}{2\sqrt[4]{b^2 - a^2}}$
$t_{kk} = cosh$
$t_{kl} = sinh$
$t_{lk} = sinh$
$t_{ll} = cosh$
end hypo

ALGORITHM 4.3. **(Jacobi for Signature Symmetric Balancing via Subsweeps)** *Given a symmetric positive definite matrix* $X \in \mathbb{R}^{n\times n}$, *an* $N = \operatorname{diag}(\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_q) \in \mathbb{R}^{n\times n}$ *with* $0 < \mu_1 < \cdots < \mu_p$ *and* $0 < \nu_1 < \cdots < \nu_q$ *and a tolerance* $\epsilon > 0$, *this algorithm overwrites* X *with* $\widetilde{X} = TXT'$, *where* $T \in \mathrm{O}^+_{pq}(\mathbb{R})$ *and* $off(\widetilde{X}) = \operatorname{tr} \widetilde{X}^2 - \sum_{i=1}^{n} \tilde{x}_{ii}^2 \leqslant \epsilon \operatorname{tr} X^2$.

$\varepsilon = \epsilon \operatorname{tr} X^2$
while $off(X) > \varepsilon$
for $k = 1 : p - 1$
for $l = k + 1 : p$
$(t_{kk}, t_{kl}, t_{lk}, t_{ll}) =$ **ortho**(X, k, l)
$X = T_{kl} X T'_{kl}$
end
end
for $k = 1 : p$
for $l = p + 1 : n$
$(t_{kk}, t_{kl}, t_{lk}, t_{ll}) =$ **hypo**(X, k, l)

$X = T_{kl} X T'_{kl}$;
end
end
for $k = p+1 : n-1$
for $l = k+1 : n$
$(t_{kk}, t_{kl}, t_{lk}, t_{ll}) = \textbf{ortho}(X, k, l)$
$X = T_{kl} X T'_{kl}$
end
end
end

5. Numerical Experiment

Now we present a numerical experiment. The initial data for the algorithm is a real positve definite symmetric $(30 \times 30)-$matrix X which is congruent to

$$\begin{aligned}\Sigma = \operatorname{diag}\,(&30, 12, 29, 6, 26, 23, 14, 11, 2, 18, 21, 19, 22, 7, 27,\\ &24, 5, 3, 1, 10, 25, 4, 13, 20, 28, 8, 17, 15, 16, 9),\end{aligned}$$

i.e., any natural number between 1 and 30 occurs as a single Hankel singular value. It holds $X = T\Sigma T'$ with $T \in \mathrm{O}^{+}_{5,25}(\mathbb{R})$. The transformation matrix $T = (t_{ij})$ is randomly generated, its squared Frobenius norm equals about $\sum_{ij} t_{ij}^2 = 130000$. The unique global fixed point

$$\begin{aligned}X_\infty = \operatorname{diag}\,(&30, 29, 26, 12, 6, 28, 27, 25, 24, 23, 22, 21, 20, 19, 18,\\ &17, 16, 15, 14, 13, 11, 10, 9, 8, 7, 5, 4, 3, 2, 1)\end{aligned}$$

is approximately reached after 5 sweeps. The accuracy of the computed Hankel singular values is about 10^{-9}. Each sweep has a computational cost of $p(p-1)/2 + q(q-1)/2 = 10 + 300 = 310$ orthogonal (2×2)-updates plus $pq = 5 \cdot 25 = 125$ hyperbolic (2×2)-updates. The figure shows the function

$$\operatorname{dist} := \operatorname{tr}\,(X_k^2 - X_\infty^2)$$

on a logarithmic scale against the number of sweeps. Plotted points are joined by linear interpolation. The squared Euclidean distance function $\operatorname{dist} = \operatorname{tr}\,(X_k^2 - X_\infty^2)$ is locally a good measure for the rate of convergence because in a neighborhood of the global minimum the manifold is locally Euclidean. Therefore, ignoring the dramatic decrease between the initial value and the value after one sweep, the decay is locally qudratically fast.

Clearly, there is a lot of room for further experiments. E.g., the ordering the subsweeps are worked off can be modified, possibly leading to improved accuracy. Questions concerning condition numbers are not raised. Also the occurence

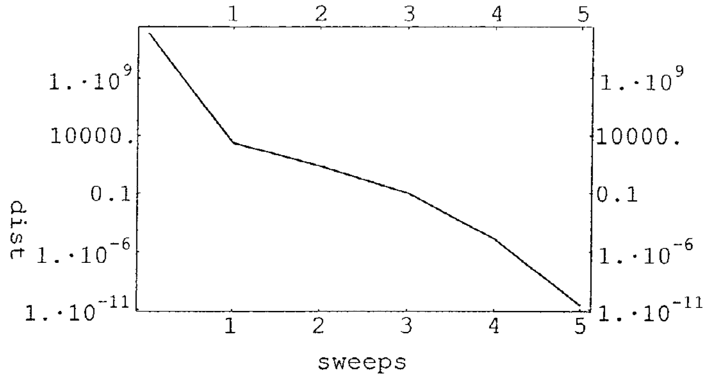

Figure 1. Balancing of an $I_{5,25}$-signature symmetric 30−th order system. dist $:= \operatorname{tr}(X_k^2 - X_\infty^2)$

of multiple and/or clustered Hankel singular values are an interesting subject to debate.

6. Conclusion

In this paper we have presented a numerical algorithm for computing balanced signature symmetric realizations. The algorithm was analyzed in detail, its convergence properties shown. One numerical example was presented verifying the theoretical results.

References

1. De Abreu-García, J.A. and Fairman, F.W. 1987, Balanced realization of orthogonally symmetric transfer function matrices. *IEEE Transactions on Circuits and Systems*, 34(9): 997–1010,
2. Anderson, B.D.O. and Bitmead, R.R. 1977, The matrix Cauchy index: Properties and applications. *SIAM J. Appl. Math.* 33: 655–672.
3. Benner, P., Quintana-Ortí, E.S. and Quintana-Ortí, G. 1999, Balanced truncation model reduction of large-scale dense systems on parallel computers. Technical report, University of Bremen, Berichte aus der Technomathematik 99-07.
4. Byrnes, C.I. and Duncan, T.W. 1982, On certain topological invariants arising in system theory. In P.J. Hilton and G.S. Young, eds., *New Directions in Applied Mathematics*, pp. 29–72. Springer, New York.
5. Fortuna, L., Nunnari, G., and Gallo, A. 1992, *Model Order Reduction Techniques with Applications in Electrical Engineering*. Springer, London.
6. Gawronski, W. 1996, *Balanced Control of Flexible Structures*. Springer, London.
7. Glover, K. 1984, All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. *Internat. J. Control*, 39(6): 1115–1193.
8. Helmke, U. and Hüper, K. 2000, A Jacobi-type method for computing balanced realizations. *Systems & Control Letters*, 39: 19–30.

9. Helmke, U. and Moore, J.B. 1994, *Optimization and Dynamical Systems*. CCES. Springer, London.
10. Hüper, K. 1996, *Structure and convergence of Jacobi-type methods for matrix computations*. PhD thesis, Technical University of Munich.
11. Laub, A.J., Heath, M.T., Paige, C.C. and Ward, R.C. 1987, Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms. *IEEE Transactions on Automatic Control*, 32(2): 115–122.
12. Liu, W.Q., Sreeram, V., and Teo. K.L. 1998, Model reduction for state-space symmetric systems. *Systems & Control Letters*, 34(4): 209–215.
13. Ober, R.J. 1987, Balanced realizations: canonical form, parametrization, model reduction. *Internat. J. Control*, 46(2): 643–670.
14. Safonov, M.G. and Chiang, R.Y. (1989), A Schur method for balanced-truncation model reduction. *IEEE Transactions on Automatic Control*, 34(7): 729–733.
15. Yan, W., Moore, J.B. and Helmke, U. 1994, Recursive algorithms for solving a class of nonlinear matrix equations with applications to certain sensitivity optimization problems. *SIAM J. Control and Optimization*, 32(6): 1559–1576.
16. Youla, D.C. and Tissi, P. 1966, N-port synthesis via reactance extraction - Part I. *IEEE Intern. Convention Record*, 183–205.